ANALYSIS OF THE APPLICABILITY OF THE ADDITIVITY LAW OF THERMAL RESISTANCES IN NONSTEADY REGIMES

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Conditions of limited applicability of the additivity law of thermal resistances in nonsteady regimes, as used in the methods of two time–temperature intervals, are determined.

1. For multilayered systems in the steady thermal regime the following additivity law of thermal resistances holds:

$$\frac{H}{\lambda'} = \sum_{i=1}^{n} \frac{h_i}{\lambda_i}, \tag{1}$$

where $H = \Sigma h_i$ is the total thickness of the multilayer system, λ_i' is the effective value of the thermal conductivity of the entire system obtained by the steady method, and h_i and λ_i are the thickness and thermal conductivity of the individual plates in the system. Under the conditions of the steady thermal regime all the methods of determining the thermal conductivity give the same value of the effective thermal conductivity for the multilayered composite systems irrespective of the order of arrangement of the plates in the system (within the admissible error).

Measuring the effective thermal conductivity of the multilayered system by different methods of the nonsteady thermal regime, we obtain different values of thermal conductivity $\lambda'' \neq \lambda'$. Furthermore, the value of the thermal conductivity λ'' determined by any one method of the nonsteady regime will be different for different arrangements of the plates in the multilayered system. This can be expressed in the following way:

$$\lambda'' = m\lambda', \tag{2}$$

where m is a correction coefficient which characterizes the degree of divergence of the values of λ'' and λ'. The quantity m depends on the nature of the nonsteady temperature field and has different values in different methods of nonsteady regimes.

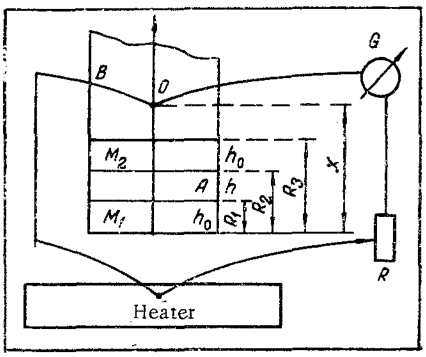

Fig. 1. A schematic diagram of the laboratory equipment.

Using the measured value of λ'' for the effective thermal conductivity of the system, we obtain an error equal to

$$\delta(\lambda') = \frac{\lambda'' - \lambda'}{\lambda'} = \frac{m\lambda' - \lambda'}{\lambda'} = m - 1. \tag{3}$$

Keeping formulas (1) and (2) in mind, we obtain

$$\sum_{i=1}^{n} \frac{h_i}{\lambda_i} = \frac{H}{\lambda'} = \frac{mH}{\lambda''}. \tag{4}$$

In formula (1), instead of the correction coefficient m we can introduce a term σ in such a way that

$$\sum_{i=1}^{n} \frac{h_i}{\lambda_i} + \sigma = \frac{H}{\lambda''}. \tag{5}$$

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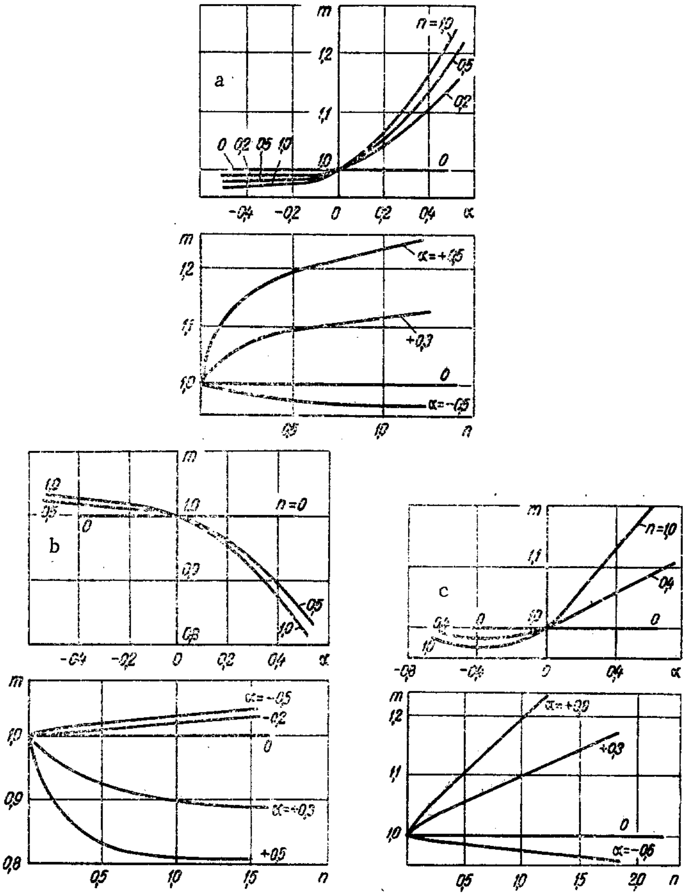

Fig. 2. Graphs of the dependence m = f(α; n): a) for θ_1 = 0.10, θ_2 = 0.25, and θ_3 = 0.50 in the first buffer method; b) for θ_1 = 0.10, θ_2 = 0.25, and θ_3 = 0.45 in the second basic method; c) for θ_1 = 0.10, θ_2 = 0.25, and θ_3 = 0.50 in the second buffer method.

The quantities m and σ are connected to each other by the formula $m = 1-(\sigma\lambda'')/H$. If the value of m or σ is known or can be determined by some calibration, then, measuring λ", Eqs. (4) or (5) can be used for determining any unknown λ for remaining unknowns λ_{i-1}.

In order to determine the justification for the applicability of the additivity law expressed by formula (1) under conditions of some nonsteady regime, it is necessary to analyze the numerical values of m or σ as a function of the characteristics of the temperature field used. This can be done if the characteristics of the temperature field used by any nonsteady method are known. In the methods of two time–temperature intervals [1] the characteristics of the temperature field are known and hence the dependence of the correction coefficient, for example, m, on certain characteristics of the field can be investigated.

The schematic diagram of the laboratory equipment is shown in Fig. 1.

2. In the first buffer method a single buffer plate M_2 which is of the same material as the heat receiver (see Fig. 1) is used. The temperature field on which the first buffer method is based is given by

$$\theta = \frac{t}{t_n} = (1+\alpha)\times\{\operatorname{erfc}[y(n+1)] - \alpha\operatorname{erfc}[y(n+3)] + \ldots\} = F(\alpha,\ n,\ y), \tag{6}$$

If the buffer plate M_2, as well as M_1, is absent, that is, if $h_0 = 0$ and n = 0, then Eq. (6) goes over to the equation of the temperature field of the first fundamental method,

$$\theta = (1+\alpha)\{\operatorname{erfc} y - \alpha\operatorname{erfc} 3y + \ldots\} = F(\alpha,\ y). \tag{7}$$

If we consider formula (4), then in the first buffer method the additivity law of thermal resistances is written in the form

$$\frac{mH}{\lambda''} = \frac{h}{\lambda} + \frac{h_0}{\lambda_0}. \tag{8}$$

Since in all the methods of two time–temperature intervals the coefficient of thermal conductivity is given by the formula $\lambda = b\varepsilon\sqrt{a}$, where

$$\sqrt{a} = \frac{h}{2\sqrt{p}\sqrt{\Delta\tau_1}} \quad \text{or} \quad \lambda = \frac{bh}{2\sqrt{\Delta\tau_1}\left(\frac{\sqrt{p}}{\varepsilon}\right)}, \tag{9}$$

the left-hand side of Eq. (8) can be expressed in the form

$$\frac{mH}{\lambda''} = \frac{2m\sqrt{\Delta\tau_1}}{b} \cdot \frac{\sqrt{p'}}{\varepsilon'}.$$

Here p' and ε' are the effective values of the parameters p and ε which are taken from operating tables of the first basic method [1]:

$$p = f_1(\kappa) = f_1\left(\frac{\Delta\tau_2}{\Delta\tau_1}\right) \text{and } \varepsilon = f_2(\kappa) = f_2\left(\frac{\Delta\tau_2}{\Delta\tau_1}\right).$$

The quantities $\Delta\tau_1$ and $\Delta\tau_2$ are determined from experiments in which the plates A and M_2 (see Fig. 1) are regarded as one investigated sample of thickness $H = h + h_0$. Next

$$\frac{h}{\lambda} = \frac{2\sqrt{\Delta\tau_1}}{b}\left(\frac{\sqrt{p}}{\varepsilon}\right),$$

where the values of parameters p and ε are related by Eq. (6) of the first buffer method. Then Eq. (8) becomes

$$\frac{2m\sqrt{\Delta\tau_1}}{b}\left(\frac{\sqrt{p'}}{\varepsilon'}\right) - \frac{2\sqrt{\Delta\tau_1}}{b}\left(\frac{\sqrt{p}}{\varepsilon}\right) = \frac{h_0}{\lambda_0}$$

or, dividing both parts by $2\Delta\tau_1/b$, we obtain

$$m\left(\frac{\sqrt{p'}}{\varepsilon'}\right) - \frac{\sqrt{p}}{\varepsilon} = \frac{h_0 b}{\lambda_0 2\sqrt{\Delta\tau_1}}.$$

Since $b = \lambda_B/\sqrt{a_B} = \lambda_0/\sqrt{a_0}$; $h_0/2\sqrt{a_0} = L$ and $L/\sqrt{\Delta\tau_1} = \xi$, then

$$m\left(\frac{\sqrt{p'}}{\varepsilon'}\right) - \left(\frac{\sqrt{p}}{\varepsilon}\right) = \xi, \tag{10}$$

and hence we obtain

$$m = \frac{\frac{\sqrt{p}}{\varepsilon} + \xi}{\frac{\sqrt{p'}}{\varepsilon'}}. \tag{11}$$

Equation (6) on which the first buffer method is based, enables us to relate the quantities ξ, $\sqrt{p}/\varepsilon$, and k for a number of values of α and n for fixed values of θ_1, θ_2, and θ_3. Knowing ξ, $\sqrt{p}/\varepsilon$, and k, and finding the quantity $\sqrt{p'}/\varepsilon'$ from the operating table of the first basic method corresponding to a given k, it is not difficult to compute the correction factor m for a number of values of α and n from Eq. (11).

The graphs of the dependence $m = f(\alpha, n)$ are shown in Fig. 2a.

3. In the second basic method the buffer plate M_2 is not there. The material of plate M_1 is the same as that of the heat receiver B (see Fig. 1).

The temperature field on which the second basic method is based is described by the equation

$$\theta = (1-\alpha^2)\{\operatorname{erfc}[y(n+1)] + \alpha \operatorname{erfc}[y(3n+1)] - \alpha \operatorname{erfc}[3y(n+1)] + \ldots\} = F(\alpha;\ y;\ n). \tag{12}$$

The notation is the same as in Eq. (6). Equation (12) also permits us to determine the quantities ξ, $\sqrt{p}/\varepsilon$, and $\sqrt{p'}/\varepsilon'$ for a number of values of α and n for fixed values of θ_1, θ_2, and θ_3, and to compute the correction factor m from formula (11) (Fig. 2b).

4. In the second buffer method, the buffer plates M_1 and M_2 are of the same thickness and consist of the same material as the heat receiver (see Fig. 1).

The temperature field on which the second buffer method is based, is described by

$$\theta = (1-\alpha^2)\{\operatorname{erfc}[y(2n+1)] + \alpha \operatorname{erfc}[y(4n+1)] - \alpha \operatorname{erfc}[y(4n+3)] + \ldots\} = F(\alpha;\ n;\ y). \tag{13}$$

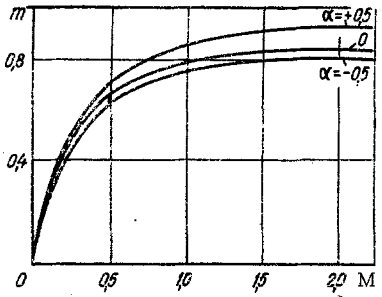

Fig. 3. Graphs of $m = f(\alpha;\, M)$ for $\theta_1 = 0.05$, $\theta_2 = 0.10$, and $\theta_3 = 0.25$ in the third method.

In this method Eq. (4) has the form

$$\frac{mH}{\lambda''} = \frac{h}{\lambda} + \frac{2h_0}{\lambda_0}$$

or

$$m\,\frac{\sqrt{p'}}{\varepsilon'} = \frac{\sqrt{p}}{\varepsilon} + 2\xi$$

and instead of (11) we have

$$m = \frac{\dfrac{\sqrt{p}}{\varepsilon} + 2\xi}{\dfrac{\sqrt{p'}}{\varepsilon'}}\,. \tag{14}$$

The graphs of $m = f(\alpha;\, n)$ are shown in Fig. 2c.

5. In the third method metallic plates M_1 and M_2 are used. The temperature field on which the third method is based is described by

$$\theta = (1 + \alpha)\,[F_1(M;\ \alpha;\ y) + F_2(M;\ \alpha;\ y) + \ldots + \ldots F_n(M;\ \alpha;\ y) + \ldots] = f(M;\ \alpha;\ y). \tag{15}$$

Equation (15) represents an infinite series in which each term is, in turn, an infinite series depending on the three arguments M, α, and y. Here, besides the notation used earlier, we have introduced the following notations: $M = h/2\gamma\sqrt{a}$ and $\gamma = (c\rho)_m h_m/b$, where $(c\rho)_m$ is the bulk specific heat of the metal from which the metallic plates are made, and h_m is the thickness of the metallic plates; $M = \eta\sqrt{p}$, where $\eta = \sqrt{\Delta\tau_1}/\gamma$. The additivity law of thermal resistances for this method has the form

$$\frac{H}{\lambda'} = \frac{mH}{\lambda''} = \frac{h}{\lambda} + \frac{2h_m}{\lambda_m}\,,$$

or

$$\frac{m\sqrt{p'}}{\varepsilon'} - \frac{\sqrt{p}}{\varepsilon} = \frac{2h_m b}{\lambda_m 2\sqrt{\Delta\tau_1}}\,.$$

Considering that $\lambda_m/b\sqrt{a_m} = \varepsilon_m$ and $h_m = \gamma b/(c\rho)_m = \gamma b a_m/\lambda_m$ we get

$$\frac{m\sqrt{p'}}{\varepsilon'} - \frac{\sqrt{p}}{\varepsilon} = \frac{1}{\varepsilon_m^2\eta}\,, \quad \text{where} \quad m = \frac{\eta\left(\dfrac{\sqrt{p}}{\varepsilon}\right) + \dfrac{1}{\varepsilon_m^2}}{\eta(\sqrt{p'}/\varepsilon')}\,. \tag{16}$$

Equation (15) permits us to relate the quantities $\sqrt{p}/\varepsilon$, $M = \eta\sqrt{p}$, and k, and, hence, the quantities $\sqrt{p}/\varepsilon$, M, and $\sqrt{p'}/\varepsilon'$ for a number of values of the parameters α and M for fixed values of θ_1, θ_2, and θ_3. We note that Eq. (16) contains another characteristic ε_m of the metal from which the metallic plates are made. The quantity $1/\varepsilon_m^2 = b^2a_m/\lambda_m^2$ is usually very small. Thus for copper plates and for a heat receiver made of plastic we have

$$\frac{1}{\varepsilon_m^2} = \frac{b^2a_m}{\lambda_m^2} = \frac{560^2\cdot 1.12\cdot 10^{-4}}{388^2} = 2.3\cdot 10^{-4} \cong 0.0002.$$

The accuracy of the computation of this quantity is much smaller than that of the quantities $\sqrt{p}/\varepsilon$, M, and $\sqrt{p'}/\varepsilon'$. If in (16) we disregard the term $1/\varepsilon_m^2$, we get

$$m = \frac{\sqrt{p}/\varepsilon}{\sqrt{p'}/\varepsilon'}\,. \tag{17}$$

The graphs of $m = f(\alpha;\, M)$ are shown in Fig. 3.

6. In the fourth method the plates M_1 and M_2 have the same thickness but the material is different from that of the heat receiver.

In this case the temperature field already depends on four parameters:

$$\theta = \frac{t}{t_n} = F(\varepsilon;\ n;\ y;\ \varepsilon_0), \tag{18}$$

TABLE 1. Results of Verification of the Applicability of the Additivity Law of Thermal Conductivity in Nonsteady Regimes

Method and investigated material	Thermal conductivity W/m·deg				
	λ	λ_x	λ'	λ''	m
First buffer method A – Vaseline oil M – plastic; $h = 2\cdot10^{-3}$ m B – plastic	0,126	0,123	0,134	0,131	0,98
Second basic method A – Vaseline oil M – plastic; $h = 2\cdot10^{-3}$ m B – plastic; $h = 0.9\cdot10^{-3}$ m	0,126	0,125	0,137	0,138	1,01
Second buffer method A – Vaseline oil M – plastic; $h_0 = 2\cdot10^{-3}$ m B – plastic	0,127	0,124	0,142	0,139	0,98
Third method A – resin; $h = 5\cdot10^{-3}$ m M – copper; $h_0 = 1.1\ 10^{-3}$ m B – quartz sand	0,170	0,140	0,207	0,171	0,82
First buffer method A – quartz glass M – protective resin $h = 2\cdot10^{-3}$ m B – protective resin; $h_0 = 2.65\cdot10^{-3}$ m	1,35	1,75	0,442	0,463	1,05
Second basic method A – quartz glass M – protective resin; $h = 2\cdot10^{-3}$ m B – protective resin; $h_0 = 2.65\cdot10^{-3}$ m	1,35	0,965	0,442	0,420	0,95

where besides the notation used above, we have introduced the additional notation $\varepsilon_0 = \lambda_0/b\sqrt{a_0}$. Unlike the second buffer method, here $\lambda_0/\sqrt{a_0} \neq \lambda_B/\sqrt{a_B}$. The additivity law of thermal resistances for this method is of the form

$$\frac{mH}{\lambda''} = \frac{h}{\lambda} + \frac{2h_0}{\lambda_0} \tag{19}$$

or

$$\frac{m\sqrt{p'}}{\varepsilon'} - \frac{\sqrt{p}}{\varepsilon} = \frac{2h_0 b}{\lambda_0 2\sqrt{\Delta\tau_1}}.$$

Multiplying and dividing the right-hand side by $\sqrt{a_0}$ we obtain $2h_0b\sqrt{a_0}/\lambda_0 2\sqrt{\Delta\tau_1}\sqrt{a_0} = 2\xi/\varepsilon_0$ and then we have

$$m = \frac{\frac{\sqrt{p}}{\varepsilon} + \frac{2\xi}{\varepsilon_0}}{\sqrt{p'}/\varepsilon'}. \tag{20}$$

An analysis of the dependence $m = f(\varepsilon, \varepsilon_0, n)$ and the construction of the corresponding graphs is the subject of a separate article, since the combination of the thermophysical properties of the heat receiver, the buffer plates, and the investigated material can be very diverse.

7. It is evident from Fig. 2 that for $\alpha \le 0$, that is, $\varepsilon \le 1$, the additivity law of the thermal resistances for the first buffer method, the second basic method, and the second buffer method is valid with an error not exceeding 2-3%.

For $\alpha > 0$, the error $\delta(\lambda') = (m-1)$ becomes appreciable. The smaller the value n and the closer the value of α to zero the smaller is the error $\delta(\lambda')$. For larger values of α the error $\delta(\lambda')$ is large even for small values of n. Thus, for example, in the first buffer method for $\alpha = +0.5$ the error $\delta(\lambda')$ reaches 11% even for n = 0.1 (see Fig. 2a).

In order to have the possibility of applying the additivity law of thermal resistances under the conditions of the third method, it is necessary that m be close to unity, that is, the quantity $\sqrt{p}/\varepsilon$ be close to $\sqrt{p'}/\varepsilon'$ as follows from formula (17). However, it is evident from Fig. 3 that this requirement is poorly satisfied. The error from the application of the additivity law in the conditions of the third method especially for small values of M can reach 50% and more.

Thus, the accuracy of determining the coefficient of thermal conductivity λ from the additivity law of thermal resistances is determined by the nature of the temperature field and the relation of the thermophysical characteristics of the investigated medium and heat receiver. For certain temperature fields the error in the determination of the coefficient λ from (1) can have large values as, for example, in the conditions of the third method. Furthermore, it should be noted that since $m = f(\alpha, n)$, the error $\delta(\lambda') = m-1$ depends on the unknown characteristics of investigated material

$$\varepsilon = \frac{1+\alpha}{1-\alpha} = \frac{\lambda}{b\sqrt{a}} \text{ and } n = \frac{h_0}{h}\sqrt{\frac{a_0}{a}}.$$

Therefore, the correction factor m (or σ) cannot be determined by any calibration method.

The coefficient of thermal conductivity λ has a clear physical meaning only for homogeneous isotropic bodies. The solid materials often used in practice have layered or fibrous structures with inclusions of holes containing air and moisture. In this case the coefficient of thermal conductivity of the same material in mutually opposite directions, determined by the nonsteady methods, will have different values. This must be taken into consideration in thermophysical investigations, in particular, in the methods of electrothermal analogies.

We noted that in the first buffer method the buffer plate is in contact with the heat receiver and the investigated medium is in contact with heater. On the other hand, in the second basic method the investigated medium is in contact with the heat receiver and the buffer plate (bottom of the vessel) is in contact with the heater. Thus an interchange of the plates M and A (see Fig. 1) is equivalent to the transition from the first buffer method to the second basic method. But the values of m for these methods with other conditions remaining equal are different (see Figs. 2a, b). This proves the validity of the statement that the effective value of the thermal conductivity of the system of contacting plates or of layered materials in conditions of nonsteady regimes depends on the order of arrangement of the plates in the multilayered system and, hence, on the direction of the heat flux.

For an experimental verification of the applicability of the additivity law of thermal resistances in conditions of nonsteady regimes we made use of those observations presented by Volkenshtein [1]. If the observations illustrating the first and second buffer methods, the second basic method, and the third method are analyzed by the first basic method, then we obtain the effective values of the thermophysical characteristics λ'' and a'' for the system of plates. Knowing the effective values λ'' and using the additivity law of thermal resistances we can calculate the value of the thermal conductivity of the investigated medium λ_x and compare it with the value λ given by some method directly. Furthermore, using the values of λ as the true value of the thermal conductivity of the investigated medium we can find the effective value λ' in the conditions of the steady regime and determine the value $m = \lambda''/\lambda'$ for the given experiment.

The results of this computation are given in Table 1. These results are in good agreement with the theoretical analysis discussed above.

LITERATURE CITED

1. V. S. Volkenshtein, High Speed Method of Determining Thermophysical Characteristics of Materials [in Russian], Énergiya, Leningrad (1971).